Noncommutative Exponential Maps on Quantum Spheres

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Quantum spheres form an important class of *noncommutative spaces*, whose coordinate functions do not satisfy the commutativity rule of multiplication. They come from the study of quantum groups and noncommutative geometry (see [1], [2], and [3]), which are active research areas in mathematics.

In particular quantum sphere S_q^2 is a deformation of the ordinary sphere S^2. To study S_q^2 it is sufficient to study its "ring of functions" $\mathcal{A}(S_q^2)$, which is a noncommutative algebra over $\mathbb{C}$ generated by three elements which subject to certain relations depending on the quantum parameter q. Moreover we can define the bundle of anti-holomorphic 1-forms $\Omega^{0,1}(S_q^2)$ as a left-right bimodule over $\mathcal{A}(S_q^2)$ and the operator $\bar{\partial} : \mathcal{A}(S_q^2) \to \Omega^{0,1}(S_q^2)$. See [4].

In the summer 2025-REU we will work on the following

Project. *For a given $f \in \mathcal{A}(S_q^2)$, find a $g \in \mathcal{A}(S_q^2)$ such that*

$$\bar{\partial} g = g \bar{\partial} f. \tag{1}$$

In the commutative case ($q = 1$), the solution is obvious since we can take $g = \exp(f) = \sum_{n=0}^{\infty} \frac{f^n}{n!}$. However in general this formula does not work since the ring is noncommutative and in particular

$$f \cdot \bar{\partial} f \neq \bar{\partial} f \cdot f.$$

We call the solution g in the general case the noncommutative exponential of f. The solutions of Equation (1) contain significant information about the structure of the quantum sphere S_q^2. Few result is known for the solutions of (1) so far. Symbolic computation softwares will help us in this project.

The project will require knowledge on calculus, linear algebra, and abstract algebra. In addition, some knowledge in complex variable functions will be helpful. On the other hand, knowledge in physics such as quantum mechanics is not required.

Activities under this project:

- The faculty will teach the student researchers on
  a) Basics of noncommutative algebras, modules, and derivations on them;
  b) Structures and properties of $\mathcal{A}(S_q^2)$;
  c) Symbolic computation software such as Mathematica and GAP.
- The student researchers will
  a) Compute the $\bar{\partial}$-operator for some elements other than the generators;
  b) Try to solve the exponential equation (1) for some element f;

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c) In the case that we cannot find a solution for (1) for a given f, try to prove that the solution does not exist;

d) The outcome of the research is open: either we can find solutions of (1) for all f, or we can find examples of f such that (1) has no solution are interesting results.

- During the last week, student researchers will give a talk on their achievements during the REU period.

- The student researcher who works on this project will:

  a) Get a good knowledge of noncommutative geometry, which is an active area of research;

  b) Understand how to work on open problems in real research;

  c) Learn how to use computer softwares to assist researches in pure mathematics.

References

[1] Edwin Beggs and S. Paul Smith. Non-commutative complex differential geometry. *J. Geom. Phys.*, 72:7–33, 2013.

[2] Edwin J. Beggs and Shahn Majid. *Quantum Riemannian geometry*, volume 355 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Cham, [2020] ©2020.

[3] Alain Connes. *Noncommutative geometry*. Academic Press, Inc., San Diego, CA, 1994.

[4] Masoud Khalkhali, Giovanni Landi, and Walter Daniël van Suijlekom. Holomorphic structures on the quantum projective line. *Int. Math. Res. Not. IMRN*, (4):851–884, 2011.